Creating a Database of Belyĭ Maps

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Abstract

A Belyĭ map $\beta : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is a rational function with at most three critical values; we may assume these values are $\{0,\, 1,\, \infty\}$. A Dessin d'Enfant is a planar bipartite graph obtained by considering the preimage of a path between two of these critical values, usually taken to be the line segment from 0 to 1. Such graphs can be drawn on the sphere by composing with stereographic projection: $\beta^{-1}\big([0,1]\big) \subseteq \mathbb{P}^1(\mathbb{C}) \simeq S^2(\mathbb{R})$. Replacing $\mathbb{P}^1$ with an elliptic curve E, there is a similar definition of a Belyĭ map $\beta : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$. Since $E(\mathbb{C}) \simeq \mathbb{T}^2(\mathbb{R})$ is a torus, we call (E, β) a toroidal Belyĭ pair. The corresponding Dessin d'Enfant can be drawn on the torus by composing with an elliptic logarithm: $\beta^{-1}\big([0,1]\big) \subseteq E(\mathbb{C}) \simeq \mathbb{T}^2(\mathbb{R})$.

This project seeks to create a database of (i) Belyĭ pairs (X, β) for either $X = \mathbb{P}^1(\mathbb{C}) \simeq S^2(\mathbb{R})$ or $X = E(\mathbb{C}) \simeq \mathbb{T}^2(\mathbb{R})$, (ii) their corresponding Dessins d'Enfant, and (iii) their monodromy groups. For each positive integer N, there are only finitely many Belyĭ pairs with $\deg \beta = N$. Using the Hurwitz Genus formula, we can begin this database by considering all possible degree sequences $\mathcal{D}$ on the ramification indices as multisets on three partitions of N. For each degree sequence, we compute all possible monodromy groups $G = \operatorname{im}\big[\pi_1(\mathbb{P}^1(\mathbb{C}) - \{0,\, 1,\, \infty\}) \to S_N\big]$. Finally, for each possible monodromy group, we compute explicit formulas for Belyĭ maps $\beta : X \to \mathbb{P}^1(\mathbb{C})$ associated to the aforementioned Riemann surfaces X. We will discuss some of the challenges of determining the structure of these groups.

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Belyĭ Pairs

Let X be a **compact, connected Riemann surface** of genus g. There are two examples of interest.

- The projective line $\mathbb{P}^1$ may be embedded into the projective plane using the map $\mathbb{P}^1 \to \mathbb{P}^2$ which sends $(x_1 : x_0) \mapsto (x_1 : 0 : x_0)$, so that this curve corresponds to the zeroes of the polynomial $f(x, y) = y$. The set of complex points, namely $X = \mathbb{P}^1(\mathbb{C}) \simeq S^2(\mathbb{R})$, is a sphere; it has genus $g = 0$.
- An elliptic curve E is a nonsingular projective variety of genus $g = 1$. There is a unique complex number $j(E) = 1728\, J$ such that E corresponds to the zeroes of the polynomial

$$f(x,y) = \begin{cases} x^3 + 1 - y^2 & \text{when } j(E) = 0; \\ x^3 - x - y^2 & \text{when } j(E) = 1728; \text{ and} \\ x^3 + \dfrac{3\,J}{1-J}\, x + \dfrac{2\,J}{1-J} - y^2 & \text{when } j(E) \neq 0,\ 1728. \end{cases}$$

The set of complex points, namely $X = E(\mathbb{C}) \simeq \mathbb{T}^2(\mathbb{R})$, is a torus.

Since X may be viewed as the set of zeroes of a single polynomial $f(x, y)$, a non-constant meromorphic function $\beta : X \to \mathbb{P}^1(\mathbb{C})$ can be written as $\beta(x, y) = p(x, y)/q(x, y)$, the ratio of two polynomials. Given P in the inverse image

$$\beta^{-1}(\omega) = \left\{ (x : y : 1) \in \mathbb{P}^2(\mathbb{C}) \;\middle|\; \begin{array}{r} f(x,y) = 0 \\ \omega_0\, p(x,y) - \omega_1\, q(x,y) = 0 \end{array} \right\}$$

we define e_P as the multiplicity of the root P in the polynomial equations above. Usually $e_P = 1$; we say $\omega \in \mathbb{P}^1(\mathbb{C})$ is a branch point if $e_P \neq 1$ for some $P \in \beta^{-1}(\omega)$. We say (X, β) is a **Belyĭ Pair** if β is unbranched away from $\{0,\, 1,\, \infty\}$.

Dessins d'Enfants

Consider the line segment $[0, 1]$ connecting 0 to 1 in $\mathbb{P}^1(\mathbb{C})$. For any Belyĭ pair (X, β), the inverse image $\beta^{-1}\big([0, 1]\big) \subseteq X$ yields a bipartite graph $\Gamma = (V, E)$ as follows:

- denote the "black" vertices as the inverse image $B = \beta^{-1}(0)$,
- the "white" vertices as the inverse image $W = \beta^{-1}(1)$,
- the edges as the inverse image $E = \beta^{-1}\big([0, 1]\big)$, and
- the midpoints of the faces as the inverse image $F = \beta^{-1}(\infty)$.

That graph with vertices $V = B \cup W$ and edges E is called a **Dessin d'Enfant**.

Monodromy Groups

The **Degree Sequence** of a Belyĭ pair is a multiset

$$\mathcal{D} = \Big\{ \{e_P \,|\, P \in B\},\ \{e_P \,|\, P \in W\},\ \{e_P \,|\, P \in F\} \Big\}$$

where $B = \beta^{-1}(0)$, $W = \beta^{-1}(1)$, and $F = \beta^{-1}(\infty)$. The Riemann-Roch Theorem asserts that

$$N = \sum_{P \in B} e_P = \sum_{P \in W} e_P = \sum_{P \in F} e_P = |B| + |W| + |F| + \big(2\,g - 2\big).$$

In particular, $\mathcal{D}$ is a multiset of three partitions of $N = \deg \beta$ into a total of N parts.

Conversely, such a collection of multisets $\mathcal{D}$ is the degree sequence for some Belyĭ pair (X, β) with $\deg \beta = N$ if and only if there exist permutations $\sigma_0,\, \sigma_1,\, \sigma_\infty \in S_N$ such that

- Each of these permutations is a product of disjoint cycles with corresponding cycle types;
- $\sigma_0 \circ \sigma_1 \circ \sigma_\infty = 1$; and
- $G = \langle \sigma_0,\, \sigma_1,\, \sigma_\infty \rangle$ is transitive subgroup of S_N.

The transitive subgroup $G = \langle \sigma_0,\, \sigma_1,\, \sigma_\infty \rangle$ is called a **Monodromy Group**.

Examples of Monodromy Groups

Such a group G may not be unique to the degree sequence. For example, $\mathcal{D} = \big\{\{1,\, 4\},\, \{1,\, 4\},\, \{5\}\big\}$ for $N = 5$ corresponds to both

$$\begin{array}{llll} & \sigma_0 = (2)\ (1\ 3\ 5\ 4) & & \sigma_0 = (2)\ (1\ 4\ 3\ 5) \\ & \sigma_1 = (4)\ (1\ 3\ 5\ 2) & & \sigma_1 = (4)\ (1\ 4\ 5\ 2) \\ & \sigma_\infty = (1\ 2\ 3\ 4\ 5) & & \sigma_\infty = (1\ 2\ 3\ 4\ 5) \\ \implies & G \simeq S_5 & \implies & G \simeq F_{20} \simeq Z_5 \rtimes Z_4 \end{array}$$

Similarly, such a group may not exist. For example, $\mathcal{D} = \big\{\{1,\, 1,\, 2,\, 2\}, \{6\}, \{6\}\big\}$ for $N = 6$ has no such group.

Algorithm

We perform the following steps.

#1. Fix a positive integer N and a nonnegative integer g. This will serve as the degree of the desired Belyĭ maps $\beta : X \to \mathbb{P}^1(\mathbb{C})$ as well as the genus of the compact, connected Riemann surface X.

#2. Compute all degree sequences $\mathcal{D}$. They are multisets of three partitions of N satisfying $N = |B| + |W| + |F| + \big(2\,g - 2\big)$.

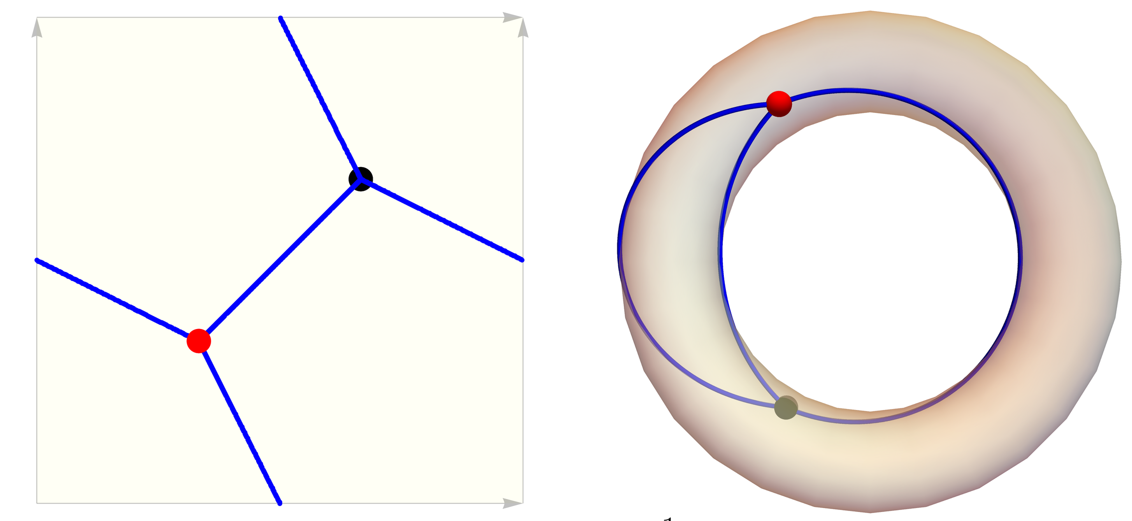
#3. For each degree sequence $\mathcal{D}$, compute all possible monodromy groups $G = \langle \sigma_0,\, \sigma_1,\, \sigma_\infty \rangle$. This amounts to searching for certain permutations $\sigma_0,\, \sigma_1,\, \sigma_\infty \in S_N$ – which we may do up to a certain equivalence relation.

#4. For each monodromy group G, compute all possible Belyĭ pairs (X, β). We know exactly how many there are by counting certain double cosets $\big|C_G(\sigma_0)\backslash G / C_G(\sigma_1)\big|$ using the Cauchy-Frobenius Lemma.

#5. For each Belyĭ pais (X, β), draw its Dessin d'Enfant.

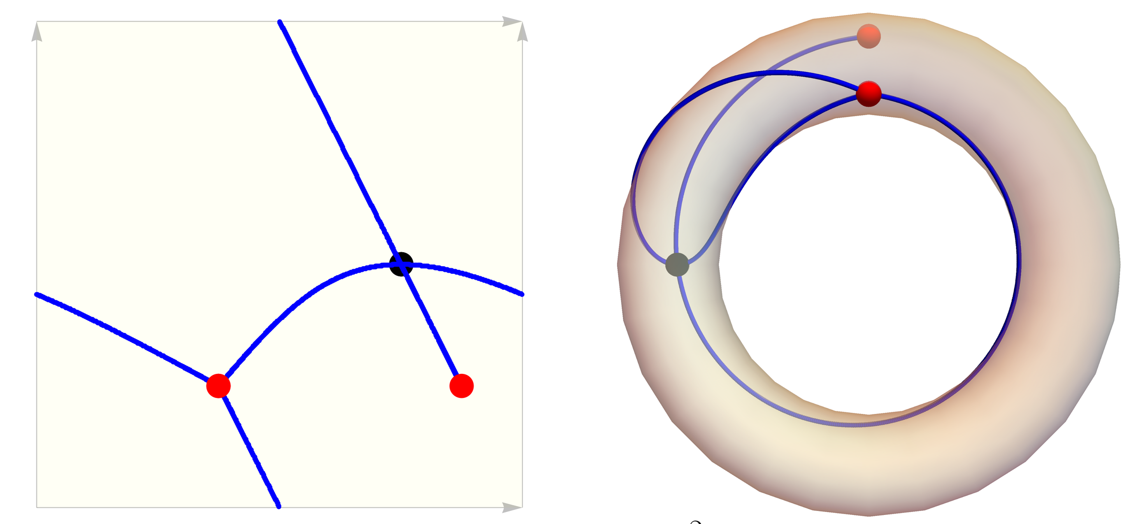
Examples of Belyĭ Pairs

- For the elliptic curve $E :\ y^2 = x^3 + 1$, the Belyĭ map $\beta(x, y) = \big(y + 1\big)/2$ has $\deg \beta = 3$ and degree sequence $\mathcal{D} = \big\{\{3\},\, \{3\},\, \{3\}\big\}$.



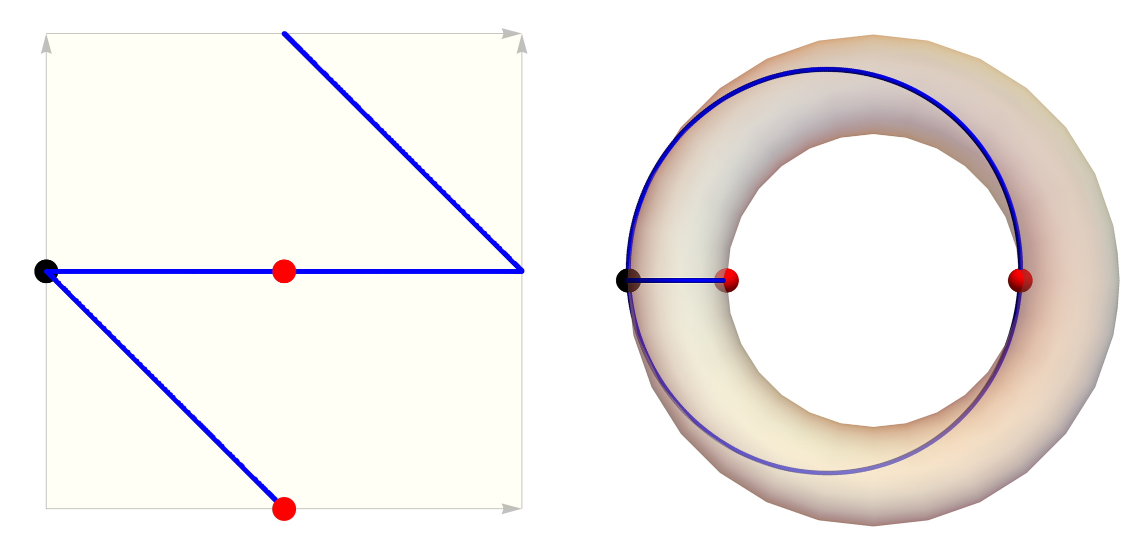
Dessin d'Enfant of $\beta(x, y) = \dfrac{y + 1}{2}$ for $E :\ y^2 = x^3 + 1$

- For the elliptic curve $E :\ y^2 = x^3 + x^2 + 16\,x + 180$, the Belyĭ map $\beta(x, y) = \big(x^2 + 4\,y + 56\big)/108$ has $\deg \beta = 4$ and degree sequence $\mathcal{D} = \big\{\{1, 3\},\, \{4\},\, \{4\}\big\}$.



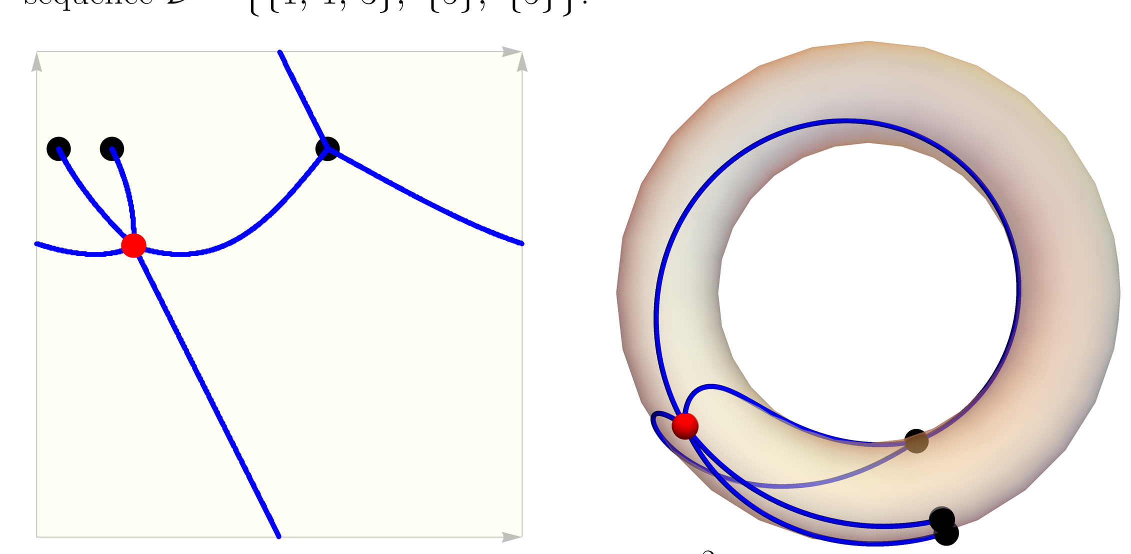
Dessin d'Enfant of $\beta(x, y) = \dfrac{x^2 + 4\,y + 56}{108}$ for $E :\ y^2 = x^3 + x^2 + 16\,x + 180$

- For the elliptic curve $E :\ y^2 = x^3 - x$, the Belyĭ map $\beta(x, y) = x^2$ has $\deg \beta = 4$ and degree sequence $\mathcal{D} = \big\{\{2, 2\},\, \{4\},\, \{4\}\big\}$.



Dessin d'Enfant of $\beta(x, y) = x^2$ for $E :\ y^2 = x^3 - x$

- For the elliptic curve $E :\ y^2 + y = x^3 + x^2 + 2\,x + 4$, the Belyĭ map $\beta(x, y) = \big(x\,y - 5\,x^2 + 7\,y - 2\,x + 15\big)/27$ has $\deg \beta = 5$ and degree sequence $\mathcal{D} = \big\{\{1,\, 1,\, 3\},\, \{5\},\, \{5\}\big\}$.



Dessin d'Enfant of $\beta(x, y) = \dfrac{x\,y - 5\,x^2 + 7\,y - 2\,x + 15}{27}$ for $E :\ y^2 + y = x^3 + x^2 + 2\,x + 4$

Results

We have only applied the algorithm for $g = 1$ and $N \leq 6$; there are 29 such sequences. We only have a handful of examples of Belyĭ maps $\beta : X \to \mathbb{P}^1(\mathbb{C})$.

Degree N	Degree Sequences $\mathcal{D}$		
$N = 1$	None		
$N = 2$	None		
$N = 3$	$\{\{3\}, \{3\}, \{3\}\}$		
$N = 4$	$\{\{1,3\}, \{4\}, \{4\}\}$		
	$\{\{2,2\}, \{4\}, \{4\}\}$		
$N = 5$	$\{\{1, 1, 3\}, \{5\}, \{5\}\}$		
	$\{\{1, 2, 2\}, \{5\}, \{5\}\}$		
	$\{\{1, 4\}, \{1, 4\}, \{5\}\}$		
	$\{\{2, 3\}, \{2, 3\}, \{5\}\}$		
	$\{\{2, 3\}, \{1, 4\}, \{5\}\}$		
$N = 6$	$\{\{1, 1, 2, 2\}, \{6\}, \{6\}\}$	$\{\{1, 1, 4\}, \{2, 4\}, \{6\}\}$	$\{\{3, 3\}, \{2, 4\}, \{2, 4\}\}$
	$\{\{1, 1, 1, 3\}, \{6\}, \{6\}\}$	$\{\{2, 2, 2\}, \{1, 5\}, \{6\}\}$	$\{\{3, 3\}, \{2, 4\}, \{1, 5\}\}$
	$\{\{2, 2, 2\}, \{3, 3\}, \{6\}\}$	$\{\{1, 2, 3\}, \{1, 5\}, \{6\}\}$	$\{\{3, 3\}, \{1, 5\}, \{1, 5\}\}$
	$\{\{1, 2, 3\}, \{3, 3\}, \{6\}\}$	$\{\{1, 1, 4\}, \{1, 5\}, \{6\}\}$	$\{\{2, 4\}, \{2, 4\}, \{2, 4\}\}$
	$\{\{1, 1, 4\}, \{3, 3\}, \{6\}\}$	$\{\{3, 3\}, \{3, 3\}, \{3, 3\}\}$	$\{\{2, 4\}, \{2, 4\}, \{1, 5\}\}$
	$\{\{2, 2, 2\}, \{2, 4\}, \{6\}\}$	$\{\{3, 3\}, \{3, 3\}, \{2, 4\}\}$	$\{\{2, 4\}, \{1, 5\}, \{1, 5\}\}$
	$\{\{1, 2, 3\}, \{2, 4\}, \{6\}\}$	$\{\{3, 3\}, \{3, 3\}, \{1, 5\}\}$	$\{\{1, 5\}, \{1, 5\}, \{1, 5\}\}$

Future Work

It will be relatively easy to create a database which consists of degree sequences $\mathcal{D}$, monodromy groups $G = \langle \sigma_0,\, \sigma_1,\, \sigma_\infty \rangle$, and the number of Belyĭ pairs (X, β) for $N \leq 20$. Computing the actual Belyĭ maps $\beta : X \to \mathbb{P}^1(\mathbb{C})$ will be extremely difficult, although drawing their Dessins d'Enfant will be easy.

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